Robustness of Fuzzy Logic Control for an Uncertain Dynamic System

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Abstract—Based on the similarity between prevalent fuzzy logic controllers (FLC) and the conventional robust controller, i.e., the variable structure controller, control theoretic analysis of a fuzzy control system is presented in the sense of Lyapunov. As well as the robustness of the fuzzy control system against uncertainties of a controlled process, this analysis gives an account of the relationship between control performance and the design parameters of the FLC, which has been obscure in the theory of fuzzy control.

Index Terms—Robustness, sliding mode control, uniform ultimate boundedness.

I. INTRODUCTION

As the fuzzy logic controller (FLC) has widened its applicability to many engineering fields, it is increasing the need of theoretic analysis such as stability and robustness to clarify the control performance of the fuzzy control system. Since there exist relatively small amount of works on the control theoretic analysis, one often hesitates to use the FLC in critical environments.

In recent years, there have been some attempts to design an FLC and explain its performance based on the theory of variable structure system. As such attempts, Kawaji et al. designed an FLC for a servomotor and tried to explain the robustness of the FLC [1], [2]. It was shown by Palm that an FLC could be regarded as an extension of conventional variable structure controller with boundary layer [3]. Also in similar way of Palm’s, Hwang et al. proposed a fuzzy sliding mode controller and opened a way of design an FLC for higher order nonlinear system [4], [5]. All these attempts are motivated by the similarity between the control rule table of prevalent FLC’s shown in Table I [6]–[9] and the output of an ordinary variable structure controller as depicted in Fig. 1. As shown in Table I, those fuzzy control rules have a same pattern all together, i.e., upper-diagonal-negative lower-diagonal-positive (UNLP) pattern.

Since the variable structure control can be applied very well to a nonlinear dynamic system with uncertainty, these attempts seem to be appropriate to show the inherent robustness of an FLC. However, even if an FLC is designed under guidance of the theory of variable structure system as in [3] and [4], the mathematical similarity between an FLC and the variable structure controller are still obscure and the closed-loop performance of fuzzy control system is not clarified yet.

It is intended in this paper to characterize the FLC with the UNLP control rules more definitely and to analyze the robustness of fuzzy control system in the sense of Lyapunov. As a result, the relationship between tracking performance and design parameters of the FLC is presented, which accounts for conventional practice in design of an FLC. In Section II, a brief introduction of an FLC and its parameterization as a kind of variable structure controller is presented and the control theoretic analysis on the robustness of a fuzzy control system is presented in Section III. As a numerical example, a design procedure and simulation results of the FLC for a simple uncertain dynamic system are presented in Section IV.

Finally, some discussions on the extensibility of the proposed
II. PARAMETERIZATION OF AN FLC

Structure of the FLC is shown in Fig. 2, of which control rules have the UNLP pattern as tabulated in Fig. 1(a). The physical meaning of the rules are as follows. For example, the control rule on the last element of Fig. 1(a) means, “Although the system error is positive big, the controller output can be zero if the derivative of error is negative big now.” It is reasonable since the system error is decreasing by itself from the negative derivative of error at that situation. From the physical meanings, those control rules are most frequently used in many fuzzy control algorithms [6]–[12].

To describe the proposed FLC more precisely, it is assumed that all input–output (I/O) fuzzy sets are designed on the normalized spaces \( e_n, \dot{e}_n, k_n \in [0, 1] \) which imply the normalized error, derivative of error, and controller output, respectively. And the I/O gains \( G_e, G_{\dot{e}}, \) and \( G_k \) serve as scale factors between the normalized spaces and the corresponding real spaces of a controlled process, i.e.,

\[
\begin{align*}
    e_n &= G_e e, \\
    \dot{e}_n &= G_{\dot{e}} \dot{e}, \quad \text{and} \quad k_f = G_k k_n.
\end{align*}
\]

A point \( c_A \) in the normalized space denotes the mean value at which the membership value of a fuzzy set \( A \) is given by one, i.e., \( \mu_A(c_A) = 1 \) where \( \mu_A(c) \) denotes the membership function of a fuzzy set \( A \). Therefore, natural choices of the mean values for exploiting the full region of the normalized spaces are as follows:

\[
\begin{align*}
    c_{\text{NBE}}(=1) &< c_{\text{NSE}} < c_{\text{ZRE}} < c_{\text{PSE}} < c_{\text{PBE}}(=1) \\
    \text{in} &\quad e_n \\
    c_{\text{NBDE}}(=1) &< c_{\text{NSDE}} < c_{\text{ZRDE}} < c_{\text{PDE}} < c_{\text{PBE}}(=1) \\
    \text{in} &\quad \dot{e}_n \\
    c_{\text{NBK}}(=1) &< c_{\text{NSK}} < c_{\text{ZRK}} < c_{\text{PSK}} < c_{\text{PBS}}(=1) \\
    \text{in} &\quad k_n.
\end{align*}
\]

Note that the following pairs of the mean values \( \{c_{\text{NBE}}, c_{\text{PBE}}\} \) and \( \{c_{\text{NBK}}, c_{\text{PBE}}\} \) in Fig. 3(a) determines a diagonal line

\[
\dot{e}_n + e_n = 0 \tag{3}
\]

in the normalized error-phase plane and (3) corresponds to the following diagonal line in the real error-phase plane from (1)

\[
\dot{e} + \sigma e = 0, \quad \sigma = \frac{G_e}{G_{\dot{e}}} > 0. \tag{4}
\]

Note also that (4) is a stable error dynamics with a decay constant \( \sigma \). Then, the following parameter \( s \)

\[
s = \dot{e} + \sigma e \tag{5}
\]

is defined as an error measure of the closed-loop control system.

From the pattern of the control rules in Fig. 1(a), the following properties on the FLC can be inferred qualitatively in the error-phase plane [3]: Generally speaking:

1) \( k_f(e, \dot{e})|_{e=0} \approx 0; \)
2) \( k_f(e, \dot{e})|_{e>0} < 0 \) and \( k_f(e, \dot{e})|_{e<0} > 0; \)
3) \( |k_f(e, \dot{e})| \propto |s|. \)

Additionally, the mapping \( k_f(e, \dot{e}) \) is continuous with respect to its arguments \( e \) and \( \dot{e} \), which is important to guarantee the existence of solution trajectory of the closed-loop control system. The continuity of input membership functions, reasoning method, and defuzzification method for continuity of the mapping \( k_f(e, \dot{e}) \) is necessary. Note that the triangular membership function, shown in Fig. 3(a), the “max–min” reasoning method, and the center-of-gravity defuzzification method are continuous and are most frequently used in many literatures.

The above properties imply the similarity between the FLC and an ordinary variable structure controller with boundary layer, which can be described in more rigorous way by the following theorem.
Theorem 1—Existence of $\phi$. For any $k_\alpha \in [0, G_k]$, there exists a positive constant $\phi$ such that the output of the FLC $k_f(\epsilon, \hat{\epsilon})$ satisfies

$$k_f = -k_\beta \frac{s}{|s|} \quad \text{in} \quad |s| \geq \phi$$

(6)

where $k_\beta > k_\alpha$ and $s = \hat{\epsilon} + \sigma e$, $\sigma = G_c/G_t > 0$.

Proof: Recall the above properties 1)–3) and the continuity of the mapping $k_f(\epsilon, \hat{\epsilon})$. Then, consider a straight line $s\perp$, which is orthogonal to $s = 0$, as shown in Fig. 4. From observation of Fig. 4(a) and (b) together with (2), it is clear that, at the two end points $p_a$ and $p_c$ and a cross point $p_b$ on $s\perp$, the output of the FLC are as follows:

$$k_f(\epsilon, \hat{\epsilon})|_{p_a} = G_k e_{NBK} = -G_k$$
$$k_f(\epsilon, \hat{\epsilon})|_{p_c} = G_k e_{ZRK} \approx 0$$
$$k_f(\epsilon, \hat{\epsilon})|_{p_b} = G_k e_{PBK} = G_k.$$  

(7)

Since $k_f(\epsilon, \hat{\epsilon})$ is continuous from the well-known mean-value theorem, for any $k_\alpha \in [0, G_k]$, there exist two points $\phi_a^*$ and $\phi_c^*$ on $s\perp$ such that

$$k_f(\epsilon, \hat{\epsilon}) < -k_\alpha \text{ in } s > \phi_a^*$$
$$k_f(\epsilon, \hat{\epsilon}) > k_\alpha \text{ in } s < \phi_c^*.$$  

(8)

These arguments are applicable to all $s\perp$.

As a consequence of $\phi = \max_{s\perp} \{|\phi_a^*|, |\phi_c^*|\}$, $k_f(\epsilon, \hat{\epsilon})$ is given as follows on the entire error-phase plane:

$$k_f(\epsilon, \hat{\epsilon}) < -k_\alpha \text{ in } s > \phi$$
$$k_f(\epsilon, \hat{\epsilon}) > k_\alpha \text{ in } s < -\phi.$$  

(9)

and it can be rewritten as (6).

Note that in this theorem, there is not any restriction on the FLC except that the control rules have the UNLP pattern. Additionally, if the mean values of the I/O fuzzy sets satisfy the following (10) and (11) as shown in Fig. 3(a)

$$c_{ZRE} = 0, \ c_{PSE} = -c_{NSE}, \ c_{PBE} = -c_{NBK} = 1$$
$$c_{ZRDE} = 0, \ c_{PSDE} = -c_{NSDE}, \ c_{PDBE} = -c_{NBDE} = 1$$
$$c_{ZRE} = 0, \ c_{PSK} = -c_{NSK}, \ c_{PBK} = -c_{NBK} = 1$$

(10)

$$c_{PSE} = c_{PSDE}, \ c_{NSE} = c_{NSDE}.$$  

(11)
then the magnitudes of $k_\alpha$ and $\phi$ in Theorem 1 can be described in terms of the design parameters of the FLC as follows.

**Corollary 1—Magnitude of $\phi$ and $k_\alpha$:** If $c_{PSK} < \frac{1}{2} c_{PBK}$, then for $k_\alpha = G_k c_{PSK}$, $k_f$ can be parameterized with respect to $\phi = 2 c_{PSK}/G_\xi$ and $k_\beta \geq k_\alpha$ as

$$k_f = -k_\beta \frac{s}{|s|} \text{ in } |s| \geq \phi.$$  

**Proof:** From the linear relationship (1) between the real spaces and the normalized spaces, this corollary is applicable to the normalized space. The normalized error phase plane shown in Fig. 5, which corresponds to the real error-phase plane in Fig. 4(b). From a little exhaustive examination, two sets of points $\phi_n^*$ and $-\phi_n^*$ are extracted (as shown in Fig. 5) by bold lines at which $k_{n_f}(\epsilon_n, \hat{\epsilon}_n)$ is given by

$$k_{n_f}(\epsilon_n, \hat{\epsilon}_n) = c_{NSK} \text{ on } s_n = \phi_n^*$$
$$k_{n_f}(\epsilon_n, \hat{\epsilon}_n) = c_{PSK} \text{ on } s_n = -\phi_n^*$$  

where $s_n = \hat{\epsilon}_n + c_n$. Here, the conditions (10) and (11) and the shapes of I/O fuzzy sets shown in Fig. 3 are considered. In addition, the reasoning and defuzzification methods are max–min and center-of-gravity, respectively. From the symmetry of the membership shapes in (10) and (11), the set of points $\phi_n^*$ and $-\phi_n^*$ are symmetric with respect to $s_n = 0$.

By applying similar arguments as the proof of Theorem 1, $k_{n_f}(\epsilon_n, \hat{\epsilon}_n)$ is written as

$$k_{n_f}(\epsilon_n, \hat{\epsilon}_n) < c_{NSK} \text{ in } s_n > \phi_n$$
$$k_{n_f}(\epsilon_n, \hat{\epsilon}_n) > c_{PSK} \text{ in } s_n < -\phi_n$$  

(14)

where $\phi_n \triangleq \max\{[\phi_n^*, -\phi_n^*]\}$ and it is given by $2c_{PSK}$. As a consequence of the relationship (1), the above (14) corresponds to

$$k_f(\epsilon, \hat{\epsilon}) < G_k c_{NSK} \text{ in } s > \phi$$
$$k_f(\epsilon, \hat{\epsilon}) > G_k c_{PSK} \text{ in } s < -\phi$$  

(15)

where $\phi = 2c_{PSK}/G_\xi$. It can be rewritten as (12).

Note from the above corollary, the boundary $\phi$ can be set arbitrarily by adjusting the design parameter of the FLC.

### III. Robustness of the Fuzzy Control System

The FLC described in Section II is applicable to the following single-input second-order uncertain dynamic system:

$$\ddot{x} = f(x) + \Delta(t) + u$$  

(16)

where $x$ denotes $[x, \dot{x}]^T$ and $\Delta(t)$ represents unstructured uncertainties such as unmodeled dynamics or external disturbances. This (16) represents various kinds of machinery dynamics and aerodynamics. Extensibility of the FLC to higher order system will be discussed in later section.

For a given bounded smooth trajectory $x_d$, $c \triangleq x - x_d$ denotes a tracking error. Using prior knowledge on $f(x)$, let the control input be

$$u = k_f(\epsilon, \hat{\epsilon}) - f_n(x)$$  

(17)

where $f_n(x)$ is known nominal dynamics of $f(x)$ and $k_f(\epsilon, \hat{\epsilon})$ is the output of the FLC.

In order to utilize the parameterization (6) of the FLC, the fuzzy control system is transformed with respect to the parameter $s$ in (5) as follows:

$$\ddot{\epsilon} = -\sigma \epsilon + s$$
$$\dddot{s} = \dddot{x} - \dddot{x}_d + \sigma \dddot{\epsilon}$$
$$= f(x) + \Delta(t) - f_n(x) + k_f(\epsilon, \hat{\epsilon}) - \dddot{x}_d + \sigma \dddot{\epsilon}$$
$$= -\eta s + F(x, t) + k_f(\epsilon, \hat{\epsilon}) + \eta s, \quad \eta > 0$$  

(18)

where $F(x, t) \triangleq f(x) - f_n(x) + \Delta(t) - \dddot{x}_d + \sigma \dddot{\epsilon}$.

In order to discuss the robustness of the closed-loop system, the ultimate boundedness of a dynamic system is useful, which implies that in spite of uncertainty, the state of a system enters eventually and remains in a small region near the equilibrium point in the state space. Thus, it describes the robustness of the system to the uncertainty [13], [14].

**Theorem 2—Uniform Ultimate Boundedness of the Fuzzy Control System:** Consider the system (18) and the parameterization (6) of the FLC. Let the output gain of the FLC be

$$G_k \geq G_{\max} + \eta |s|, \quad \eta > 0$$  

(19)

where $G_{\max} \triangleq \max_{x \in x} |f(x) - f_n(x)| + \max_{t |\Delta(t)| + |\dddot{x}_d| + \sigma |\dddot{\epsilon}|}$. Then the closed-loop fuzzy control system is ultimately bounded.
Fig. 5. Examination of a set of $\phi_n^*$ in the normalized error-phase plane.

Proof: To prove the theorem, (20) is chosen as a Lyapunov function candidate of the closed-loop control system

$$V(y) = y^tPy$$

(20)

where $P = \begin{bmatrix} \eta \sigma & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $y \triangleq [e, s]^t$.

Time derivative of (20) along the solution trajectory of (18) is

$$\dot{V}(y) = \begin{bmatrix} e_s \end{bmatrix}^t 2P \begin{bmatrix} e \\ s \end{bmatrix} + \begin{bmatrix} e_s \end{bmatrix} 2P \begin{bmatrix} 0 & -1 \\ \sigma & 0 \end{bmatrix} \begin{bmatrix} e \\ s \end{bmatrix} + \begin{bmatrix} e_s \end{bmatrix} 2P \begin{bmatrix} 0 \\ F(x, t) + \alpha \phi + \eta \phi + G_k \end{bmatrix} = -y^tAy + s(F(x, t) + \alpha \phi + \eta \phi + G_k)$$

(21)

where $A = \begin{bmatrix} 2\eta \sigma & -2\eta \sigma \\ 0 & \frac{1}{2} \end{bmatrix}$ is a positive definite matrix. Recall that for $k_{\alpha} \geq F_{\max} + \eta \phi + k_{\alpha}$, there exists $\phi$ which parameterizes the FLC as (6) from Theorem 1.

Thus, if $[s] \geq \phi$ as a consequence of $k_{\beta} \geq k_{\alpha}$ and $F_{\max} \geq [F(x, t)]$, (21) becomes

$$\dot{V}(y) = -y^tAy + s(F(x, t) + \eta \phi + k_{\alpha}) \leq -y^tAy + s(F(x, t) + \eta \phi + k_{\alpha}) \leq -y^tAy + s(F(x, t) + \eta \phi + k_{\alpha})$$

(22)

where $\lambda_{\min}(A)$ is the minimum eigenvalue of $A$.

Then, consider the other case of $[s] < \phi$. From (21), together with $[k_{\beta}(e, \dot{e})] \leq G_k$, $V(e, \dot{e}) \in \mathbb{R}^2$, $\dot{V}$ becomes

$$\dot{V}(y) \leq -y^tAy + \phi(F_{\max} + \eta \phi + G_k) \leq -y^tAy + \phi(F_{\max} + \eta \phi + G_k)$$

(23)

where $0 < \rho < 1$ and

$$\delta = \{y|(1 - \rho)\lambda_{\min}(A)|y|^2 \leq \phi(F_{\max} + \eta \phi + G_k)\}.$$ (24)

As the result of both cases, $\dot{V}$ is given by

$$\dot{V}(y) \leq -\lambda_{\min}(A)|y|^2, \quad \forall |y| \geq \delta(\phi)$$

(25)

which is illustrated in Fig. 6 and (25) implies the uniform ultimate boundedness of the closed-loop control system as defined in [13] and [14].

Remark: The ultimate bound $\delta$, which implies the tracking error in the steady state, depends on the magnitude of $\phi$ as shown in Corollary 1, the relationship between the design parameters of the FLC and the magnitude of $\phi$ is given as

$$\phi = \frac{2CPSE}{G_0}$$

This result explains the following designing conception adopted by fuzzy control engineers: in order to get small steady-state error, the input fuzzy sets should be set up...
densely near an equilibrium point and coarsely far from the equilibrium point, which can be done by making $c_{PSE}$ small while keeping $c_{PBK}$ as one. Although this result is derived under some assumptions in the designing stage of the FLC as discussed in Corollary 1, those assumptions are not severe since those are generally used and can be relaxed.

### IV. A Numerical Example

In order to verify the above results in Sections II and III, the proposed FLC is applied to the Duffing forced-oscillation system (26) with uncertainty

$$\ddot{x} = -a \dot{x} - b x^3 + c \cos(t) + \Delta(t) + u$$

which is chaotic in unforced case, i.e., $u = 0$. The unforced trajectory of the system is shown in $(x, \dot{x})$ plane in Fig. 7 for $x(0) = \dot{x}(0) = 0$ and $a = 0.11$, $b = 0.9$, and $c = 12.2$.

It is assumed that intervals of structured uncertainties are known as

$$0.05 \leq a \leq 0.15, \ 0.5 \leq b \leq 1.5, \ 10.0 \leq c \leq 14.0.$$  \hfill (27)

For unstructured uncertainty $\Delta(t)$, a bounded random signal as (28) is injected intentionally.

$$-2.0 \leq \Delta(t) \leq 2.0.$$  \hfill (28)

For unstructured uncertainty $\Delta(t)$, a bounded random signal as (28) is injected intentionally. The nominal dynamics of $f(x) = -a \dot{x} - b x^3 + c \cos(t)$ is given as $f_n(x) = -0.1 \dot{x} - 1.0 x^3 + 12.0 \cos(t)$ by taking the mean values of the interval (27). Thus, the modeling error as the uncertainty is given by

$$F(x, t) = f(x) - f_n(x) + \Delta(t) - \ddot{x}_d + \sigma \dot{e}$$

$$= -(a - 0.1) \dot{x} - (b - 1.0) x^3 + (c - 12.0) \cos(t) + \Delta(t) - \ddot{x}_d + \sigma \dot{e}$$

from (27) and (28), and its upper bound can be taken as

$$F_{\text{max}} = \max_{x} |f(x) - f_n(x)| + \max_{\Delta} |\Delta(t)| + |\ddot{x}_d + \sigma \dot{e}|$$

$$= 0.05 |\dot{\theta}| + 1.0 |x^3| + 2.0 |\cos(t)| + 1.0 + |\ddot{x}_d + \sigma \dot{e}|.$$  \hfill (29)

From the table, the slope $\sigma$ of the parameter $s$ and the magnitude of $\phi$ are given as

$$\sigma = \frac{G_e}{G_e} \approx 0.667$$

and

$$\phi = 2 \frac{c_{PSE}}{G_e} \approx 0.003$$

which implies the time-constant of error dynamics in (4) and the tracking accuracy of closed-loop system, respectively.

The simulations are carried out using the control input (17) with (19) and $\eta = 2.0$. In order to investigate the regulating and tracking performances of the fuzzy control system, the

![Fig. 7. Trajectory of the Duffing forced oscillation in unforced case.](image)
following two different time functions are chosen as desired trajectories:

\[ x_d(t) = u_0(t) \text{ and } x_d(t) = \sin \left( \frac{\pi}{3,0} t \right) \]

where \( u_0(t) \) denotes the unit step function. Desired trajectories and the corresponding responses of the closed-loop system are shown in Figs. 8(a) and 9(a). Here, the initial conditions of the controlled process are \( x(0) = 0, \dot{x}(0) = 0 \) for \( x_d(t) = u_0(t) \) and \( x(0) = -0.5, \dot{x}(0) = 0 \) for \( x_d(t) = \sin \left( \frac{\pi}{3,0} t \right) \).

In these figures, the tracking errors are ultimately bounded and the steady-state errors are given by \( \varepsilon_1 \approx 1.5 \times 10^{-2} \) in the unit-step response and \( \varepsilon_2 \approx 4.0 \times 10^{-2} \) in the sinusoidal response. The magnitude of steady-state error depends on the value of \( \phi \) as mentioned before. The control inputs are shown in Figs. 8(b) and 9(b). Although there are some chattering, it can be eliminated by enlarging the magnitude of \( \phi \). Note that the fuzzy control system is robust to uncertainties since the tracking errors do not diverge in spite of the uncertainties.

V. DISCUSSIONS

In order to extend the above discussions to more general cases, the followings should be considered.

Although a controlled process are limited to a single-input second-order system as like (16), the analysis and the design guideline can be easily extended to a higher order system as follows.

1) For a higher order system, which can be decoupled to \( n \) sets of second-order systems by feedback compensation, e.g., a robot manipulator with \( n \) joints, \( n \) sets of FLC should be given for \( n \) of the second-order systems. Each FLC has same form of a control rule table and membership functions with different values of its own design parameters.

2) In case of a higher order system, which cannot be decoupled, the rule table of an FLC should have corresponding dimension with the order of the system. Recall that the UNLP form of control rule table for the second-order system is skew symmetric with respect to a diagonal line and the diagonal line constitutes a stable error dynamics \( s = \dot{e} + \sigma e, \sigma = G_e / G_k > 0 \) in the error-phase plane. Similarly, the form of control rule table for the \( n \)th order system should be skew symmetric with respect to an \( n \)-dimensional hyperplane. And the hyperplane should constitute a stable error dynamics in the \( n \)-dimensional space by choosing appropriate input gains of the FLC.
An exemplar stable hyperplane in the $n$-dimensional space is given by

$$s = \left( \frac{d}{dt} + \sigma \right)^{n-1} e = 0, \quad \sigma > 0$$  \hspace{1cm} (31)

which is critically damped [15]. In order to reduce the total number of control rules, the error measure $s$ and its derivative $\dot{s}$ can be used as the input variables of the FLC instead of $e$, $\dot{e}$, $\ldots$, and $e^{(n-1)}$ [3], [4].

3) In the case where a controlled process has the input uncertainty like

$$\dot{\phi}^* = \frac{d}{dt} (\dot{\phi} + \Delta \dot{\phi}) u$$  \hspace{1cm} (32)

the performance of the fuzzy control system can be guaranteed by the slightly modified version of the proposed method, which is well known in the theory of variable structure system [15].

VI. CONCLUDING REMARKS

Although it is increasing the applicability of an FLC to many engineering fields as an intelligent control approach recently, it seems to be very risky to use it for critical environments without guaranteed performances. Thus, endeavor to guarantee the performance of a fuzzy control system is worthy, although it is difficult since it gives the reliability to the control system as well as the better understanding of the control performances.

Motivated by observation on similarity between prevalent FLC’s with the UNLP pattern of the control rules and the conventional variable structure controller, the robustness of the FLC has been analyzed for a class of nonlinear dynamic systems with bounded uncertainties in this paper. As results, the asymptotic behavior of the fuzzy control system can be clarified and the relationship between the design parameters of the FLC and the tracking performances of the control system is addressed explicitly. The relationship is important since it gives a guidance on the design parameters of the FLC to achieve the specified control performances. In order to verify the results, a numerical example is presented for the Duffing forced-oscillation system with known bounded uncertainties and the computer simulations are carried out in which the FLC gives good tracking performances of the control system in spite of the uncertainties as expected.

Since the output gain of the FLC is dependent on the upper bound of uncertainty (as stated before), the output of resulting controller may be conservative in a sense. Thus, in order to reduce the degree of uncertainty, thereby enhancing the efficiency of the controller, it will be helpful to introduce a modeling tool for the uncertainty into the controller.

APPENDIX

EXAMINATION OF $-\phi^*$ FOR $k_{n,} = k_{PSK}$

As a consequence of (1), there is a linear relationship between $-\phi^*$ for $k_{n} = k_{CPSE}$ and $-\phi^*$ for $k_{n,} = k_{PSK}$. Thus, the examination procedure can be done in the normalized error-phase plane. Since the mapping $k_{n,} f(c_n, \ell_n)$ is so complex and nonlinear, it is helpful to divide the whole normalized error-phase plane into several subregions and examine $-\phi^*$ in each subregion. It is natural to divide the normalized error-phase plane according to the input fuzzy sets. Here, an examination procedure in a subregion $C_1 - C_2 - D_2 - D_1$ shown in Fig. 10 is presented as an example.

In the region, $C_1 - C_2 - D_2 - D_1$, the activated rules are as follows:

If $c_n$ is NBE and $\ell_n$ is ZRDE, then $k_{n,f}$ is PBK
If $c_n$ is NBE and $\ell_n$ is PSDE, then $k_{n,f}$ is PBK
If $c_n$ is NSE and $\ell_n$ is ZRDE, then $k_{n,f}$ is PSK
If $c_n$ is NSE and $\ell_n$ is PSDE, then $k_{n,f}$ is ZRK.

(33)

Since the consequent part of the rules (33) are PBK, PSK, and ZRK, the cog defuzzification gives

$$k_{n,f}(c_n, \ell_n) = \frac{CPBK + CPK + CSRK}{\mu_{PBK} + \mu_{PSK} + \mu_{ZRK}}$$

$$= \frac{CPBK + CPK + CSRK}{\mu_{PBK} + \mu_{PSK} + \mu_{ZRK}}$$

(34)
From the triangular shape of fuzzy sets shown in Fig. 10, the following relationships between membership values hold in this region

\[
\mu_{NB}(e_n) = 1 - \mu_{NS}(e_n),
\mu_{PS}(\hat{e}_n) = 1 - \mu_{ZR}(\hat{e}_n),
\]  

(35)

Thus, the possibilities of the activated output fuzzy sets are as follows:

\[
\mu_{PBK} = \max\{\min(\mu_{NB}; \mu_{ZR}); \min(\mu_{NB}; \mu_{PS})\} = \max\{\min(1 - \mu_{NSE}; 1 - \mu_{PS}); \min(1 - \mu_{NSE}; \mu_{PS})\}
\]

\[
\mu_{PSK} = \min(\mu_{NSE}; \mu_{ZR})
\]

\[
\mu_{ZRK} = \min(\mu_{NSE}; \mu_{PS}).
\]

(36)

According to dominant membership values, this region is divided into four subregions, as shown in Fig. 10. It is noted that as a consequence of (10) and (11) and the triangular fuzzy sets, the relationships \(\mu_{PS} = 1 - \mu_{NSE}\) and \(\mu_{PS} = \mu_{NSE}\) hold on line segments \(D_1 - C_2\) and \(C_1 - D_2\), respectively.

1) Region I: \(\mu_{NSE} < \mu_{PS} < 1\). From (36) and the relationships between the membership values in this region, the possibilities of the activated output fuzzy sets are as follows:

\[
\mu_{PBK} = \max\{1 - \mu_{PS}; \mu_{PS}\}
\]

\[
\mu_{PSK} = \mu_{NSE}
\]

\[
\mu_{ZRK} = \mu_{NSE}.
\]

(37)

By inserting (37) into (34), \(k_n, f(e_n, \hat{e}_n)\) is given by

\[
k_n, f(e_n, \hat{e}_n) = CPBK \cdot \max\{1 - \mu_{PS}; \mu_{PS}\} + CPSK \cdot \mu_{NSE}
\]

\[
\times \max\{1 - \mu_{PS}; \mu_{PS}\} + 2 \mu_{NSE},
\]

(38)

From elementary mathematics, it can be shown that this \(k_n, f(e_n, \hat{e}_n)\) achieves a minimum at the point \(p = (\gamma e_n, \mu_{NSE}(e_n) = \frac{1}{2}, \mu_{PS}(\hat{e}_n) = \frac{1}{2}\) and its minimum value is

\[
k_n, f(e_n, \hat{e}_n)|_p = \frac{CPBK + CPSK}{3},
\]

(39)

Note that the condition \(CP_{PS} < \frac{1}{2}CPBK\) in Corollary 1 can be written as

\[
CP_{PS} = \frac{1}{2 + \alpha} CPBK, \quad \alpha > 0.
\]

(40)

Inserting (40) into (39) gives

\[
k_n, f(e_n, \hat{e}_n)|_p = \frac{3 + \alpha}{3} CP_{PS}.
\]

(41)

Since it is minimum in Region I, \(k_n, f(e_n, \hat{e}_n) > CP_{PS}\) holds for all \((e_n, \hat{e}_n)\) and there is no point \(-\phi_n^*\) for \(k_n, \alpha = CP_{PS}\) in this region.

2) Region III: \(\mu_{PS} < \mu_{NSE} < 1 - \mu_{NSE}\). From (36), the possibilities of the activated output fuzzy sets are as follows:

\[
\mu_{PBK} = 1 - \mu_{NSE}
\]

\[
\mu_{PSK} = 1 - \mu_{PSD}
\]

\[
\mu_{ZRK} = \mu_{PSD}.
\]

(42)

Inserting (42) into (34) gives

\[
k_n, f(e_n, \hat{e}_n) = \frac{CPBK(1 - \mu_{NSE}) + CPSK(1 - \mu_{PS})}{2 - \mu_{NSE}}.
\]

(43)

By definition of \(-\phi_n^*, k_n, f(e_n, \hat{e}_n) - \phi_n^* = CP_{PS}\) together with \(CPBK = 1\), (40), and (43), the set of \(-\phi_n^*\) in this region satisfies

\[
\mu_{PSD}(\hat{e}_n) = (1 + \alpha) - (1 + \alpha)\mu_{NSE}(e_n)
\]

(44)

which is depicted by bold line in Fig. 10.

3) Regions II and IV: By a similar manner as 1) and 2), the locus of \(-\phi_n^*\) in these subregions can be obtained and it is also shown in Fig. 10.

\[\square\]

REFERENCES


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