A Robust Fuzzy Logic Controller for Robot Manipulators with Uncertainties

Soo Yeong Yi and Myung Jin Chung

Abstract—Owing to load variation and unmodeled dynamics, a robot manipulator can be classified as a nonlinear dynamic system with structured and unstructured uncertainties. In this paper, the stability and robustness of a class of the fuzzy logic control (FLC) is investigated and a robust FLC is proposed for a robot manipulator with uncertainties. In order to show the performance of the proposed control algorithm, computer simulations are carried out on a simple two-link robot manipulator.

I. INTRODUCTION

A robot manipulator is an uncertain nonlinear dynamic system which suffers from structured and unstructured uncertainties such as load variation, friction, and external disturbances etc. Therefore, in order to control the robot manipulator, it is necessary a control algorithm having simple computation and robustness to uncertainties.

On the other hand, it is known that fuzzy logic controllers (FLC’s) have the programming capability of human control behavior and possess the robustness to uncertainties. Accordingly, there has been increasing efforts to introduce fuzzy set theory into the control of robot manipulators [1], [2]. However, there is no definite stability/robustness analysis of the closed-loop system since FLC is often designed in a heuristic manner. Although such analysis is difficult, every endeavor to guarantee the performance of a fuzzy control system is valuable.

In this paper, it is intended to propose a robust FLC for robot manipulators with uncertainties, based on the stability analysis of FLC in [3], which is motivated by the similarity between FLC and conventional robust variable structure control [4], [5]. In addition to the stability analysis of FLC, the inherent properties of the Lagrange–Euler dynamics of robot manipulators are considered to design the FLC more efficiently. Since the FLC consists of simple and fast computational elements, it is appropriate for dynamic control of robot manipulators. In order to verify the fuzzy control algorithm, computer simulations are carried out on a simple two-link robot manipulator.

II. THE PROPOSED FUZZY CONTROL ALGORITHM

The Lagrange–Euler dynamics of an n-link robot manipulator can be described as

\[
M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) + \Delta = \tau,
\]

where \(M(q)\in \mathbb{R}^{n \times n}\) is an inertia matrix, \(C(q, \dot{q})\dot{q}\in \mathbb{R}^{n}\) represents centrifugal and Coriolis forces, and \(g(q)\in \mathbb{R}^{n}\) is the vector of gravitational forces. And \(\Delta\) represents unstructured uncertainties which is upper-bounded as

\[
|\Delta_i| \leq \Delta_{\max,i}, \quad i = 1, \ldots, n.
\]

The FLC is shown in Fig. 1 of which control rules have so-called UNLP (Upper-diagonal-Negative, Lower-diagonal-Positive) pattern as tabulated in Table I. The physical meaning of the rules is as follows. For example, the control rule on the last element of Table I means, “Although the system error is Positive Big, the controller output can be ZeRo if the derivative of error is Negative Big now.”

It is well-known that the matrix

\[
N(q, \dot{q}) = M(q) - 2C(q, \dot{q})
\]

is skew-symmetric [6] and

\[
M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) + \Delta = Y(q, \dot{q}, q, \dot{q}, \theta),
\]

\[
Y \in \mathbb{R}^{n \times p}, \quad \theta \in \mathbb{R}^{p}
\]

(3)

where \(\theta\) is a constant vector of inertia parameters [7] and \(Y(q, \dot{q}, \theta)\) consists of known functions of the generalized coordinate. Using prior knowledge on the inertia parameters, the structured uncertainty and its upper bound can be represented as

\[
|\theta_{0,i} - \hat{\theta}_i| \leq \rho_i, \quad i = 1, 2, \ldots, p
\]

(4)

where \(\theta_{0,i}\) is the known nominal value of \(\hat{\theta}_i\) and \(\theta_{0,i}\) denotes the structured uncertainty. For a bounded smooth trajectory \(q, \dot{q}\), a tracking error is

\[
\epsilon = q - \hat{q}.
\]

The FLC is shown in Fig. 1 of which control rules have so-called UNLP (Upper-diagonal-Negative, Lower-diagonal-Positive) pattern as tabulated in Table I. The physical meaning of the rules is as follows. For example, the control rule on the last element of Table I means, “Although the system error is Positive Big, the controller output can be ZeRo if the derivative of error is Negative Big now.”

It is reasonable since the system error decreases by itself in that situation. From those physical meanings, these control rules are most frequently used in many fuzzy control algorithms [1], [8]–[10].

To describe the proposed FLC more precisely, all input/output fuzzy sets are assumed to be designed on the normalized space \(\epsilon_n, \dot{\epsilon}_n, \tau_n \in [-1, 1]\). And the input/output gains \(G_e, G_{\dot{e}}, \text{ and } G_k\)
serve as scale factors between the normalized space and the corresponding real space of a controlled process, i.e., \( c_n = G_x \hat{e}_n \) and \( \tau = G_y \). A point \( \xi \) in the normalized space denotes the mean value at which the membership value of a fuzzy set \( A \) is given by one, i.e., \( \mu_A(\xi) = 1 \) where \( \mu_A(\hat{\xi}) \) denotes the membership function of a fuzzy set \( A \). Therefore, natural choices of the mean values for exploiting the full region of the normalized spaces are

\[
\begin{align*}
&c_{NH}(-1) < c_{NS} < c_{Z} < c_{PS} < c_{PH} = 1 \\
&c_{ND}(-1) < c_{NS} < c_{Z} < c_{PS} < c_{PH} = 1 \\
&c_{NH}(-1) < c_{NS} < c_{Z} < c_{PS} < c_{PH} = 1 \\
&c_{NH}(-1) < c_{NS} < c_{Z} < c_{PS} < c_{PH} = 1.
\end{align*}
\]

Note that the following pairs of the mean values \( c_{NH}, c_{PH}, c_{PS}, c_{ND} \) in Fig. 2(a) determines a diagonal line

\[
\hat{e}_n + e_n = 0 \quad (6.1)
\]

in the normalized error phase plane. And it corresponds to the following diagonal line in the real error phase plane from the input/output gains of the FLC

\[
\hat{e} + \sigma e = 0, \quad \sigma = \frac{G_x}{G_y} > 0. \quad (6.2)
\]

Note also that (6.2) is a stable error dynamics. Then, a parameter \( r = \hat{e} + \sigma e \) represents as an error measure of the closed-loop control system. From the pattern of the control rules in Table 1, the following properties on the FLC can be inferred qualitatively in the error phase plane [5]. Generally speaking

1) The output is zero near a diagonal line, \( r = 0 \).
2) The output is negative above the diagonal line, \( r > 0 \).
3) The output is positive below the diagonal line, \( r < 0 \).
4) The magnitude of the output tends to increase in accordance with the distance between the diagonal line, \( r = 0 \), and the system state \((e, \hat{e})\).

These properties imply the similarity between the FLC and a conventional variable structure controller.

The following theorem describes the above properties more rigorously.

**Theorem 1 (Existence of \( \phi \) and \( k_n \)):** For any \( k_n \in [0, G_x] \), there exists a boundary \( \phi \) such that the output of the FLC, \( \tau_\phi \), satisfies

\[
\tau_\phi = -k_f \frac{r}{|r|} \text{ in } |r| \geq \phi \quad (7)
\]

where \( k_f > k_n \) and \( r = \hat{e} + \sigma e, \sigma = \frac{G_x}{G_y} > 0 \).

**Proof:** see Appendix A.

The above theorem shows that the FLC with the UNLP rule pattern can be regarded as an extension of a variable structure controller. Additionally, if the mean values of the input/output fuzzy sets satisfy

\[
\begin{align*}
c_{Z} &= 0, \quad c_{PS} = -c_{NS}, \\
c_{PH} &= -c_{NH} = 1 \\
c_{ZD} &= 0, \quad c_{PS} = -c_{NS}, \\
c_{PHD} &= -c_{ND} = 1 \\
c_{PS} &= c_{PSE}, \quad c_{NS} = c_{NDE}.
\end{align*}
\]

then the magnitudes of \( k_n \) and \( \phi \) in Theorem 1 can be described by the design parameters of the FLC as follows.
where \( v = q - \Sigma e \) and \( a = v \) [11]. Then, the following theorem describes the robustness of the fuzzy control system consisting of (11) and control input (12)

\[
\tau = -Kr + Y(q, \dot{q}, \dot{a}, a)\theta_0 + r_f \quad (12)
\]

where \( K \in \mathbb{R}^{n \times n} \) is a symmetric positive definite matrix, \( Y\theta_0 \) is a known nominal value of \( Y\theta \) in (11), and \( r_f \) is the output vector of the fuzzy controller to suppresses the structured and unstructured uncertainties \( Y(\theta - \theta_0) + \Delta \). In order to discuss the robustness of the closed-loop system, the ultimate boundedness of a dynamic system is useful which implies that, in spite of uncertainty, the state of a system enters eventually and remains in a small region near an equilibrium point in the state space. Thus it describes the robustness of the system to the uncertainty [12].

**Theorem 3 (Ultimate boundedness):** If the output gain \( G_{k,i}, i = 1, \ldots, n \) of the FLC satisfies (13), then the closed-loop system is ultimately bounded

\[
G_{k,i,FP SK,i} = [Y(q, \dot{q}, a, v)\theta_i + \Delta_{max,i}, i = 1, 2, \ldots, n \quad (13)
\]

where \( \rho = [\rho_1, \ldots, \rho_p] \) and \( |Y(q, \dot{q}, a, v)\theta_i| \) is the magnitude of the \( i \)-th element of \( Y(q, \dot{q}, a, v)\theta \in \mathbb{R}^n \).

**Proof:** Inserting (12) into (11) makes the closed-loop system as follows:

\[
\dot{X} = M(q)\dot{r} + C(q, \dot{q}) + K\dot{r} + \Delta
\]

Then, (15) is chosen as a Lyapunov function candidate for (14)

\[
V = \frac{1}{2}r^T M(q)\dot{r} + e^T \Sigma^T K e.
\]

Differentiating (15) and inserting (14) gives

\[
\dot{V} = r^T M(q)\dot{r} + \frac{1}{2}r^T M(q)\dot{r} + 2e^T \Sigma K e
\]

where \( \Sigma = \text{diag}[\Sigma^T K \Sigma, K] \). In order to investigate \( V \) in the bounded interval of \( t \), and \( L_1 \) and \( L_2 \) are defined as \( L_1 = \{ |r_i| \geq \bar{o}_i \forall i \} \) and \( L_2 = \{ |r_i| < \bar{o}_i \forall i \} \) so that \( L_1 \) and \( L_2 \) constitute a partitioning of \( \mathbb{R}^n \).

i) When \( t \in L_1 \), from (16) together with (9) and (13), \( \dot{V} \) is given by

\[
\dot{V} \leq -x^T Q x + \sum_{i=1}^{n} |r_i| |Y(q, \dot{q}, a, v)\theta_i| + \Delta_{max,i} \]

\[
+ \sum_{i=1}^{n} r_i \cdot -k_{f,i} \frac{r_i}{|r_i|} \leq -x^T Q x + \sum_{i=1}^{n} |r_i| |Y(q, \dot{q}, a, v)\theta_i| + \Delta_{max,i} - k_{f,i} \]

\[
\leq -\lambda_{min} |x|^2 + \sum_{i=1}^{n} \bar{o}_i \bar{o}_i \leq \Phi, \quad \forall i \] (17)

where \( \lambda_{min} \) is the minimum eigenvalue of the positive definite matrix \( Q \).

ii) Consider the other case of \( t \in L_2 \). Since the output fuzzy sets are defined on the normalized space \( k_{n,i} \in [-1, 1] \), it is clear that \( |r_{ij,i}| \leq G_{k,i, \bar{o}_i} \). Thus from (13) we obtain

\[
|r_{ij,i}| \leq \frac{1}{c_{FP SK,i}} |Y(q, \dot{q}, a, v)\theta_i| + \Delta_{max,i} \quad \forall i \quad (18)
\]

By inserting (18) and (13) into (16), \( \dot{V} \) can be written as

\[
\dot{V} \leq -x^T Q x + \sum_{i=1}^{n} r_i \{ |Y(q, \dot{q}, a, v)\theta_i| + \Delta \} + \sum_{i=1}^{n} r_i \cdot r_i \cdot -k_{f,i} \frac{r_i}{|r_i|} \]

\[
\leq -x^T Q x + \sum_{i=1}^{n} \bar{o}_i \{ |Y(q, \dot{q}, a, v)\theta_i| + \Delta_{max,i} \} \]

\[
+ \sum_{i=1}^{n} r_i \cdot -k_{f,i} \frac{r_i}{|r_i|} \leq -x^T Q x + \sum_{i=1}^{n} \bar{o}_i \{ |Y(q, \dot{q}, a, v)\theta_i| + \Delta_{max,i} \}
\]

\[
+ \sum_{i=1}^{n} \bar{o}_i \bar{o}_i \leq \Phi, \quad \forall i \]

(19)

As a consequence of both cases i) and ii), \( \dot{V} \) can be written as

\[
\dot{V} \leq -\lambda_{min} |x|^2, \quad \text{if } |x| \geq \Phi \quad (20)
\]

where \( \Phi \) is the worst case value of \( \Phi \) in (22) with \( l = \{ |v| = 1, \ldots, n \} \). Equation (23) implies the ultimate boundedness of the closed-loop system as defined in [12].
The Lagrange–Euler dynamics of a simple two-link robot manipulator, shown in Fig. 4, is represented by (24), shown at the bottom of the page, where \( G \) is the gravitational constant, \( C_i, S_i, C_{ij}, \) and \( S_{ij} \) denote \( \cos q_i, \sin q_i, \cos(q_i + q_j), \sin(q_i + q_j) \), \( i, j = 1, 2, \) respectively. As an unstructured uncertainty, \( \Delta \), the bounded random noise as in (27.1) is injected intentionally. From the reparameterization of (24) for the parameters in (25), each element \( y_{ij} \) of \( Y(q, \dot{q}, \ddot{q}, a) \) is given by

\[
\begin{align*}
\theta_1 &= m_1 l_1^2 + m_2 l_2^2 + I_1 \\
\theta_2 &= m_2 l_2^2 + I_2 \\
\theta_3 &= m_2 l_2 c_2 \\
y_{11} &= a_1 \\
y_{12} &= a_1 + a_2 \\
y_{13} &= \cos(q_2)(2a_1 + a_2) - \sin(q_2)(\dot{q}_2v_2 + \dot{q}_1v_1 + \dot{q}_2v_1) \\
y_{14} &= g \cos(q_1) \\
y_{15} &= g \cos(q_1) \\
y_{16} &= g \cos(q_1 + q_2) \\
y_{21} &= 0 \\
y_{22} &= a_1 + a_2 \\
y_{23} &= \cos(q_2)a_1 + \sin(q_2)a_1 v_1 \\
y_{24} &= 0 \\
y_{25} &= 0 \\
y_{26} &= g \cos(q_1 + q_2).
\end{align*}
\] (25)

From the parameters of the unloaded arm shown in Table II [11] and the interval for the structured uncertainty in (27.2), the mean values for the range of possible \( \theta_1 \) are chosen as the nominal value of the parameter \( \theta \) which are shown in Table III

\[
-25.0 \leq \Delta_1 \leq 25.0, \quad -12.5 \leq \Delta_2 \leq 12.5
\] (27.1)

\[
0 \leq \Delta l_{c2} \leq 10.0, \quad 0 \leq \Delta l_{c2} \leq 0.5, \quad 0 \leq \Delta l_{c2} \leq \frac{1}{27}
\] (27.2)

Thus, the upper bound of the structured uncertainty \( \rho \) is given as shown in Table IV.

The structure of the FLC and the membership shape of the input/output fuzzy sets are shown in Figs. 1 and 2. According to (8.1), (8.2), and Theorem 2, the design parameters of the FLC are chosen as shown in Table V.

Thus, the slope \( \sigma_i \) of \( r_i \) and the magnitude of the boundary \( \phi_i \), which imply time-constant of the desired characteristics of the tracking error and the tracking accuracy in the steady-state respectively are given as

\[
\sigma_1 = \frac{G_{c,1}}{G_{c,1}} = 2.0, \quad \sigma_2 = \frac{G_{c,2}}{G_{c,2}} = 2.0
\]

\[
\phi_1 = \frac{2 \rho_{FS1}}{G_{c,1}} = 8.0 \times 10^{-3}, \quad \phi_2 = \frac{2 \rho_{FS2}}{G_{c,2}} = 1.6 \times 10^{-2}
\] (28)

Simulations are carried out using the dynamics (24) and the control law (12) where \( K = \text{diag}[5.0, 5.0] \). A desired trajectory of the second-order critically damped model and the corresponding tracking error of each joint are shown in Fig. 5(a) and (b), respectively. As
shown in the figures, the tracking errors are ultimately bounded and steady-state errors are given by \( e_1 \approx 1.5 \times 10^{-3} \) (rad) and \( e_2 \approx 3.3 \times 10^{-3} \) (rad). The value of steady-state error depends on the value of \( \hat{\phi}_i \), as shown in Theorem 3. Thus, the tracking accuracy can be set arbitrarily by the design parameter \( C_{PS, i} \) of the FLC.

The input torque of two joints is shown in Fig. 5(c) and (d). In those figures, some chattering of the input torque is shown in steady-state. It can be reduced by enlarging \( \hat{\phi}_i \). There is a tradeoff between the magnitude of chattering and the tracking accuracy.

The responses on the following desired trajectory of a circle (29) in the Cartesian space are shown in Fig. 6.

\[
x = 0.7 + 0.5 \sin s(t) \\
y = 0.7 + 0.5 \cos s(t), \quad -\pi \leq s(t) \leq \pi, \quad 0 \leq t \leq T
\] (29)

where \( s(t) = -\pi \cos(\pi/T)t \) and \( T = 2 \) s. The desired trajectory in the Cartesian space can be transformed into the desired trajectory in the joint space by the inverse Jacobian of the manipulator which is shown in Fig. 6(a). The upper bounds on the tracking error are given by \( e_1 \approx 2.8 \times 10^{-3} \) (rad) and \( e_2 \approx 0.7 \times 10^{-3} \) (rad) as shown in Fig. 6(b). The input torque of each joint are shown in Fig. 6(c). The tracking results in the Cartesian space are depicted in Fig. 6(d) where the desired trajectory and the actual trajectory are represented by a dotted line and a solid line respectively. In the figures, the FLC shows a good tracking performance as analyzed using Theorem 3.

### IV. Conclusion

There are various sources of uncertainties in the dynamics of robot manipulator such as load variation, friction, unmodeled dynamics, and
external disturbances etc. Thus it is necessary a robust controller to overcome the uncertainties. Since FLC has been known to be robust to uncertainty, it is intended in this paper to design a robust FLC for dynamic control of uncertain robot manipulators. In order to give reliability to the fuzzy control system, the ultimate boundedness of the closed-loop control system is addressed by using the well-known robust control theory and inherent properties of robot dynamics. The analysis accounted for the relationship between the design parameters of the FLC and the tracking accuracy explicitly. Thus the tracking performance can be set arbitrarily by adjusting the design parameters of the FLC. The proposed FLC was verified by computer simulations on a simple two-link robot manipulator.

APPENDIX A
PROOF OF THEOREM 1

The following properties on the mapping \( \tau_f(e, \dot{e}) \) are obtained by considering the control rule table shown in Table I and the error measure, \( r = \hat{e} + \sigma e, \sigma = G_c / G_r \) [5].

1) \( \tau_f(e, \dot{e}) \big|_{r=0} \approx 0. \)
2) \( \tau_f(e, \dot{e}) \big|_{r>0} < 0 \) and \( \tau_f(e, \dot{e}) \big|_{r<0} > 0. \)
3) \( |\tau_f(e, \dot{e})| \propto |r|. \)
4) \( \tau_f(e, \dot{e}) \) is a continuous mapping since it consists of continuous mappings e.g., triangular membership functions of input fuzzy sets, max-min reasoning and cog defuzzification method as assumed before.

Then, consider a straight line \( s^\bot \) which is orthogonal to \( r = 0 \) as shown in Fig. 7. From observation of Fig. 7(a) and (b), it is clear that, at the two end points \( p_a \) and \( p_e \) and a cross point \( p_b \) of \( s^\bot \), the output of the FLC are

\[
\begin{align*}
\tau_f(e, \dot{e}) |_{p_a} &= G_k c_{NHK} = -G_k \\
\tau_f(e, \dot{e}) |_{p_b} &\approx G_k c_{ZHK} = 0 \\
\tau_f(e, \dot{e}) |_{p_e} &= G_k c_{PHK} = G_k.
\end{align*}
\]

Since it is continuous, from the well-known mean-value-theorem, for any \( k_{alp, a} \in [0, G_k] \), there exist points \( \phi_a^* \) and \( \phi_e^* \) on \( s^\bot \) such that

\[
\begin{align*}
\tau_f(e, \dot{e}) < -k_a & \quad \text{in } r > \phi_a^* \\
\tau_f(e, \dot{e}) > k_a & \quad \text{in } r < \phi_e^*.
\end{align*}
\]

These arguments is applicable to all \( s^\bot \). As a consequence of \( \phi = \max_{s^\bot} \{|\phi_a^*|, |\phi_e^*|\} \), \( \tau_f(e, \dot{e}) \) is given as follows on the entire error phase plane

\[
\begin{align*}
\tau_f(e, \dot{e}) < -k_a & \quad \text{in } r > \phi \\
\tau_f(e, \dot{e}) > k_a & \quad \text{in } r < -\phi.
\end{align*}
\]

Equation (A.3) can be rewritten as (7) in Theorem 1.

APPENDIX B
PROOF OF THEOREM 2

The normalized error phase plane is shown in Fig. 8 which corresponds to the real error phase plane in Fig. 7(b) from the input gains \( G_e \) and \( G_r \). Some exhaustive examinations extract a set of points \( \phi_a^* \) and \( -\phi_e^* \) for \( k_{n, a} = c_{NSK} \) and \( k_{n, a} = c_{PSK} \) as shown.
Because of the linear relationship between the real variables and the normalized variables, it is useful to divide the normalized error phase plane into several subregions and examine $\phi^*_n$ in each subregion of the normalized error phase plane. It is natural to divide the normalized error phase plane according to the input fuzzy sets. Here, an examination procedure in a subregion $C_1 - C_2 - D_2 - D_1$ shown in Fig. 9 is presented as an example.

In the region, $C_1 - C_2 - D_2 - D_1$, the activated rules are

\[
\begin{align*}
&\text{If } e_n \text{ is } NBE \text{ and } \dot{e}_n \text{ is } ZRDE, \text{ then } \tau_{n,i} \text{ is } PBK \\
&\text{If } e_n \text{ is } NBE \text{ and } \dot{e}_n \text{ is } PSDE, \text{ then } \tau_{n,i} \text{ is } PBK \\
&\text{If } e_n \text{ is } NSE \text{ and } \dot{e}_n \text{ is } ZRDE, \text{ then } \tau_{n,i} \text{ is } PSK \\
&\text{If } e_n \text{ is } NSE \text{ and } \dot{e}_n \text{ is } PSDE, \text{ then } \tau_{n,i} \text{ is } ZRK. \quad \text{(C.1)}
\end{align*}
\]

Since the consequent part of the rules (C.1) are $PBK, PSK$, and $ZRK$, the cog defuzzification gives

\[
\begin{align*}
\tau_{n,i}(e_n, \dot{e}_n) &= \frac{c_{PBK} \mu_{PBK} + c_{PSK} \mu_{PSK} + c_{ZRK} \mu_{ZRK}}{c_{PBK} + c_{PSK} + c_{ZRK}} \\
\mu_{NBE}(e_n) &= 1 - \mu_{NSEE}(e_n) \\
\mu_{PSDE}(e_n) &= 1 - \mu_{ZRDE}(e_n). \quad \text{(C.3)}
\end{align*}
\]

Thus the possibilities of the activated output fuzzy sets are

\[
\begin{align*}
\mu_{PBK} &= \max \{ \min(\mu_{NBE}, \mu_{ZRDE}), \min(\mu_{NBE}, \mu_{PSDE}) \} \\
&= \max \{ \min(1 - \mu_{NSEE}, 1 - \mu_{PSDE}) \} \\
&= \min(1 - \mu_{NSEE}, \mu_{PSDE}) \\
\mu_{PSK} &= \min(\mu_{NSEE}, \mu_{ZRDE}) \\
&= \min(\mu_{NSEE}, 1 - \mu_{PSDE}) \\
\mu_{ZRK} &= \min(\mu_{NSEE}, \mu_{PSDE}). \quad \text{(C.4)}
\end{align*}
\]

According to dominant membership values, this region is divided into four subregions as shown in Fig. 9. It is noted that, as a
Fig. 9. Examination of a set of $\phi_n^*$ in the subregion $C_1 - C_2 - D_2 - D_1$.

Consequence of (8.1) and the triangular fuzzy sets, the relationship $\mu_{PSE} = 1 - \mu_{NSE}$ on a line segment $D_1 - C_2$ and $\mu_{PSE} = \mu_{NSE}$ on a line segment $C_1 - D_2$ hold, respectively.

a) Region I: $\mu_{NSE} < \mu_{PSE} < \mu_{NSE} < \mu_{PSE} < 1 - \mu_{NSE}$

From (C.4), the possibilities of the activated output fuzzy sets are

$$\mu_{PHK} = \max\{1 - \mu_{PSE}, \mu_{PSE}\}$$

$$\mu_{PSK} = \mu_{NSE}$$

$$\mu_{ZRK} = \mu_{NSE}.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} (C.5)

By inserting (C.5) into (C.2), $\tau_{n,f}$ is given by

$$\tau_{n,f}(e_n, e_n) = \frac{cp_{PHK} \max\{1 - \mu_{PSE}, \mu_{PSE}\} + cp_{PSK}}{\max\{1 - \mu_{PSE}, \mu_{PSE}\} + 2\mu_{NSE}}. \hspace{1cm} (C.6)$$

From elementary mathematics, it can be shown that it achieves a minimum at the point $p = \{(e_n, e_n)|\mu_{NSE}(e_n) = \frac{1}{2}, \mu_{PSE}(e_n) = \frac{1}{2}\}$ and its minimum value is

$$\tau_{n,f}(e_n, e_n)|_p = \frac{cp_{PHK} + cp_{PSK}}{3}. \hspace{1cm} (C.7)$$

Note that the condition $cp_{PSK} < \frac{1}{2}cp_{PHK}$ in Theorem 2 can be parameterized as

$$cp_{PSK} = \frac{1}{2 + \alpha} cp_{PHK}, \hspace{1cm} \alpha > 0. \hspace{1cm} (C.8)$$

Inserting (C.8) into (C.7) gives

$$\tau_{n,f}(e_n, e_n)|_p = \frac{3 + \alpha}{3} cp_{PSK}. \hspace{1cm} (C.9)$$

Since it is minimum in the Region I, $\tau_{n,f} > cp_{PSK}$ $\forall (e_n, e_n)$ $\in$ Region I holds and there is no point $-\phi_n^*$ for $\bar{e}_n = cp_{PSK}$ in this region.

b) Region III: $\mu_{PSE} < \mu_{NSE} < 1 - \mu_{NSE} < \mu_{PSE}$

From (C.4), the possibilities of the activated output fuzzy sets are

$$\mu_{PHK} = 1 - \mu_{NSE}$$

$$\mu_{PSK} = 1 - \mu_{PSE}$$

$$\mu_{ZRK} = \mu_{PSE}.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} (C.10)

Inserting (C.10) into (C.2) gives

$$\tau_{n,f}(e_n, e_n) = \frac{cp_{PHK}(1 - \mu_{NSE}) + cp_{PSK}(1 - \mu_{PSE})}{2 - \mu_{NSE}}. \hspace{1cm} (C.11)$$

By definition of $-\phi_n^* |_{\bar{e}_n} = cp_{SK}$ together with $cp_{BH} = 1$, (C.6), and (C.11), the set of $-\phi_n^*$ in this region satisfies

$$\mu_{PSE}(e_n) = (1 + \alpha) - (1 + \alpha)\mu_{NSE}(e_n) \hspace{1cm} (C.12)$$

which is depicted by bold line in Fig. 9.

c) Region II and Region IV:

By similar manner as a) and b), the locus of $-\phi_n^*$ in these subregions can be obtained and it is shown in Fig. 9 also.

\[\square\]

References


