Abstract—In this note, we present a robust observer-based $H_{\infty}$ control design method for linear time-delay systems with norm-bounded time-varying uncertainties. Using the Riccati-equation-based approach we design observer-based feedback control laws, which guarantee the quadratic stability of the closed-loop control system and reduce the effect of the disturbance input on the controlled output to a prescribed level. © 1997 Elsevier Science Ltd.

1. Introduction

Time delay is commonly encountered in various engineering systems, such as chemical processes, long transmission lines in pneumatic, hydraulic, and rolling mill systems. And time delay usually results in unsatisfactory performances and is frequently a source of instability. Recently, by extending the state space $H_{\infty}$ controller design methods (Petersen, 1987; Doyle et al., 1989) several authors have proposed $H_{\infty}$ control methods for linear systems with time delay (Choi and Chung, 1995a, 1996; Ge et al., 1996a, b; and references therein). Especially, Choi and Chung (1996) propose two alternative methods for designing observer-based $H_{\infty}$ control laws whose gain matrices are obtained in terms of the solutions of a pair of Riccati-like equations. The delayed system models considered in the above methods have no parameter uncertainty. However, in most practical situations system models suffer from model parameter variation. The previous methods cannot be directly applied to robust $H_{\infty}$ stabilization of linear time-delay systems with norm-bounded time-varying uncertainties and robust $H_{\infty}$ control design methods for such systems cannot be trivially derived from the existing results.

In this note, we extend the result of Choi and Chung (1996) in order to include linear time-delay systems with norm-bounded time-varying uncertainties. We present a method for designing robust observer-based $H_{\infty}$ control laws for such systems. In terms of a pair of Riccati-like equations we give a sufficient condition for quadratic stability with $H_{\infty}$ norm-bound. Since the proposed method is an extended version of the earlier Riccati equation approaches, through some slight modification the earlier results can be reproduced. We also give a simple design algorithm for obtaining gain matrices, together with an example.

2. Model description and background results

Let the system to be controlled be represented by the following differential equation:

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]u(t) + [A_d + \Delta A_d(t)]x(t - h) + D_w(t),$$

$$y(t) = [C + \Delta C(t)]x(t) + w(t),$$

with $x(t) = x(t), t \in [-h, 0].$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $y \in \mathbb{R}^p$ is the measured output, $w_1(t) \in \mathbb{R}^r, w_2(t) \in \mathbb{R}^s$ are the square integrable disturbances, $\varepsilon \in \mathbb{R}^r$ is the controlled output, $\phi(t) \in \mathbb{R}^r$ is the continuous initial value function, and $A, B, A_d, C, D$ and $E$ are constant matrices with appropriate dimensions. $\Delta A, \Delta B, \Delta A_d$ and $\Delta C$ are real-valued matrix functions representing time-varying parameter uncertainties. It is assumed that the system uncertainties are norm-bounded of the following form:

$$\Delta A(t) = H_1 F_1(t) N_1, \quad \Delta B(t) = H_2 F_2(t) N_2,$$

$$\Delta A_d(t) = H_3 F_3(t) N_3, \quad \Delta C(t) = H_4 F_4(t) N_4,$$

where $H, N_1, N_2, N_3, N_4$ are known constant matrices with appropriate dimensions, and properly dimensioned matrix function $F(t)$ is unknown but norm-bounded as

$$F_1(t) F_2(t) \leq I.$$

Before proceeding further, we will give some preliminary results. Let us consider the following uncertain time-delay system:

$$\dot{x}(t) = A_c x(t) + [A_d + H_1 F_1(t) N_1]x(t - h) + D_w(t),$$

$$z(t) = L_c x(t).$$

where $A_c, A_d, H_1, N_1, D_w$ and $L_c$ are constant matrices with appropriate dimensions.

Definition. For some given positive constant $\gamma$, the uncertain time-delay system (2) is said to be quadratically stable with an $H_{\infty}$-norm bound $\gamma$ if the following properties (P1) and (P2) hold:

$$\text{(P1)} \exists \text{ a Lyapunov functional such that for some positive constants } \alpha, \beta,$$

$$z_t \|x(t)\|^2 \leq L(t) \leq \sup_{-h \leq t \leq 0} z_t \|x(t + \theta)\|^2. $$

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and the time derivative of the Lyapunov functional along the solution trajectory of (2) with $w(t)=0$ satisfies $\dot{V}(t) \leq -\gamma \|y(t)\|^2$ for all time. (P2) Subject to the zero initial condition $\psi(t) = 0$, the controlled output $z$ satisfies $\|z\|^2 \leq \gamma \|w\|^2$ for square-integrable disturbance input $w$, i.e.,

$$\int_0^\infty \|z(t)\|^2 \, dt \leq \gamma \int_0^\infty \|w(t)\|^2 \, dt.$$  

Using the above notion of quadratic stability with $H_{\infty}$ norm-bound, one can establish the following technical lemma.

**Lemma 1.** For some given positive constant $\gamma$ and some positive-definite weighting matrix $\Psi$, if there exists a positive-definite matrix $\bar{P}$ satisfying the following inequality:

$$\bar{A}_c^T \bar{P} + \bar{P} \bar{A}_c + \bar{P} \bar{A}_d \bar{A}_c^T + \Psi + \bar{P} \bar{H}_2 \bar{H}_1^T \bar{P} < 0,$$

then the uncertain time-delay system (2) is quadratically stable with an $H_{\infty}$ norm-bound $\gamma$.

**Proof.** From the results given in Choi and Chung (1995b), one can see that (P1) holds if there exists a solution $P > 0$ satisfying the above matrix inequality. By incorporating the results of Xie and De Souza (1992) and Choi and Chung (1996), one can also show that if the above matrix inequality is solvable then (P2) is satisfied. $\square$

### Main results

Now, consider the following output-feedback dynamic control law:

$$\dot{x}(t) = \tilde{A} \tilde{x}(t) + Bu(t) + H_2 R \tilde{x}(t) + A_0 G \tilde{x}(t) + \tilde{H} \tilde{x}(t),$$

$$u(t) = K \hat{x}(t),$$

where $\tilde{x} \in \mathbb{R}^n$ is the estimated state of $x$, the terms $H_2 R \tilde{x}(t), A_0 G \tilde{x}(t)$ and $\tilde{H} \tilde{x}(t)$ account for $\Delta Bu(t), \Delta A_0 \tilde{x}(t-h), \Delta A_0 G \tilde{x}(t-h)$ and $\Delta \tilde{H} \tilde{x}(t)$, respectively, and $L, K, F, R, S$ and $G$ are the gain matrices with appropriate dimensions. It should be noted that the above control law (3) needs instantaneous values of the estimated state and (3) corresponds to the control law of the second type given in Choi and Chung (1996). By introducing the observer error $e(t) = x(t) - \hat{x}(t)$, we get the following augmented system:

$$\dot{\tilde{x}}(t) = [\tilde{A} + \Delta \tilde{A}] \tilde{x}(t) + [\tilde{A}_d + \Delta \tilde{A}_d] \tilde{e}(t) + \Delta \tilde{w}(t),$$

where $\tilde{w}(t) = [w_1(t), w_2(t)]$, and

$$\tilde{A} = \begin{bmatrix} A + BK & -BK \\ A - A_0 \end{bmatrix}, \quad \tilde{A}_d = \begin{bmatrix} A_d \ 0 \end{bmatrix}.$$  

$$\Delta \tilde{A} = \begin{bmatrix} \Delta A + \Delta B \cdot K & -\Delta B \cdot K \\ L \cdot \Delta C + \Delta \tilde{B} \cdot K & -\Delta B - K \end{bmatrix},$$

$$\Delta \tilde{A}_d = \begin{bmatrix} \Delta A_d \ 0 \\ \Delta A_d \ 0 \end{bmatrix}.$$  

Now, we are ready to establish our main result.

**Main Theorem.** Consider the system (1) and suppose that the control parameters are given by

$$K = -\frac{1}{\varepsilon_c} \left( I + N_1^2 N_2 \right)^{-1} B^T P_c, \quad F = \frac{1}{\varepsilon_c} D^T P_c,$$

$$L = -\frac{1}{\varepsilon_c} \left( I + N_2 H_1 \right)^{-1},$$

$$G = \frac{1}{\varepsilon_c} \left( I + N_1 H_1 \right)^{-1},$$

where $\varepsilon_c, \varepsilon_0$ and $\psi_c$ are some positive constants, $P_c$ and $P_0$ are, respectively, positive-definite solution matrices to the following Riccati-like equations:

$$P_c A + A^T P_c - \frac{1}{\varepsilon_c} \varepsilon P_c \left( I + N_1^2 N_2 \right)^{-1} B^T - \frac{1}{\varepsilon_0} A_0 A_0^T \psi_0 \Psi_c,$$

$$\frac{1}{\varepsilon_c} \left[ H_1^T H_2 + H_2^T H_1 + \gamma^2 H_3 H_3^T + \gamma H_4 H_4^T \right] - \frac{1}{\varepsilon_c} \left[ H_3^T H_2 + H_2^T H_3 + \gamma H_4 H_4^T \right] = 0,$$

for a positive constant $\varepsilon$ and some positive-definite weighting matrices $Q_c$ and $Q_0$. Then, the closed-loop system is quadratically stable with an $H_{\infty}$ norm-bound $\gamma$.

**Proof.** By Lemma 1, it is sufficient to show that for some positive-definite weighting matrix $\Psi$ there exists a positive-definite matrix $P$ satisfying the following inequality:

$$P \Delta \tilde{A} + \Delta \tilde{A}^T P + \Delta \tilde{A}_d \Psi^{-1} \Delta \tilde{A}_d^T P + \Psi \left[ \begin{array}{c} H_1 \\ H_2 \\ K \end{array} \right] \left[ \begin{array}{c} H_1^T \\ H_2^T \\ K^T N_1^T N_2 \end{array} \right] \left[ \begin{array}{c} N_3 \\ 0 \\ 0 \end{array} \right] = 0.$$

One can easily show that the following inequality is true:

$$P \Delta \tilde{A} + \Delta \tilde{A}^T P \leq P \left[ \begin{array}{c} H_1 \\ H_2 \\ K \end{array} \right] \left[ \begin{array}{c} H_1^T \\ H_2^T \\ K^T N_1^T N_2 \end{array} \right] \left[ \begin{array}{c} N_3 \\ 0 \\ 0 \end{array} \right] + P \left[ \begin{array}{c} H_1 \\ H_2 \\ K \end{array} \right] \left[ \begin{array}{c} H_1^T \\ H_2^T \\ K^T N_1^T N_2 \end{array} \right] \left[ \begin{array}{c} N_3 \\ 0 \\ 0 \end{array} \right].$$

And, if we define $P$ and $\Psi$ as

$$P = \left[ \begin{array}{cc} \psi_0 P_c & 0 \\ 0 & \gamma^2 \psi_0^2 \end{array} \right], \quad \Psi = \left[ \begin{array}{cc} \psi_c I & 0 \\ 0 & \psi_0 I \end{array} \right],$$

where $P_c$ and $P_0$ are, respectively, positive-definite solution matrices to the Riccati-like equations (7). Then
through some routine algebraic manipulations one can show that \( P \) satisfies
\[
P(\hat{A} + \Delta \hat{A}) + (\hat{A} + \Delta \hat{A})^T P + P \hat{A} \Psi^{-1} \hat{A}^T P + \Psi
\]
\[
+ P \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix} \begin{bmatrix} N_1^T & 0 \\ 0 & N_2 \end{bmatrix} + \begin{bmatrix} N_1^T & 0 \\ 0 & N_2 \end{bmatrix} [N_1, 0]
\]
\[
+ \frac{1}{\gamma^2} \Psi \Psi^T [Q_0, 0] \leq 0.
\]
After all, one can conclude that the closed-loop system quadratically stable with an \( H_\infty \)-norm bound \( \gamma \).

Some remarks are in order.

**Remark 1.** It should be noted that one may use the following output feedback control law instead of (3) and derive similar sufficient conditions
\[
\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + H_2 \Phi(t) + A\Delta x(t - h) + H_3 \Sigma(t) + I(C\hat{x}(t) - y(t)) + DF\hat{x}(t),
\]
\[
\dot{u}(t) = K\hat{x}(t).
\]
The above control law corresponds to the first type given in Choi and Chung (1996) while (3) corresponds to the second. As discussed in Choi and Chung (1996), the above control law is more difficult to implement from a practical point of view than (3) because the above requires past values of the estimated state while (3) requires only instantaneous values.

**Remark 2.** \( \hat{A}_d \) of (5a) can be rewritten as:
\[
\hat{A}_d = \begin{bmatrix} A_d \\ A_d \end{bmatrix} [I \ 0]
\]
and, therefore, the terms corresponding to \( \frac{1}{\gamma^2} \Psi \Psi^T A_d A_d^T \) can be rewritten as
\[
\frac{1}{\gamma^2} \Psi \Psi^T \left[ \begin{bmatrix} A_d^T \\ A_d \end{bmatrix} \right] P \phi + \psi \phi \left[ \begin{bmatrix} I \\ 0 \end{bmatrix} \right] \phi^T.
\]
After all, through some routine algebraic manipulations one can show that Main Theorem is still valid, even though the Riccati-like equation (7a) is replaced by
\[
A_d P_o + P_o A_d^T = -I_{60} [C_1^T (I + \gamma^2 H_1 H_1^T)^{-1} C
\]
\[
+ \frac{1}{\gamma^2} K^T (I + N_1 N_1^T) K] P_o
\]
\[
+ \psi \phi \left[ \begin{bmatrix} I \\ 0 \end{bmatrix} \right] \phi^T.
\]
Along the same line of the arguments in Choi and Chung (1996) we can establish some statements relating to the existence as well as the uniqueness of the solution matrices of the given Riccati-like equations.

**Remark 3.** Suppose that there exists a constant \( \epsilon > 0 \) such that equation (7a) has a positive-definite solution matrix \( P \), for some given constants \( \psi \), \( \gamma \), and for some given positive-definite weighting matrix \( Q_o \). And suppose that the following matrix inequality holds:
\[
B(I + N_1 N_1^T)^{-1} B^T - \frac{1}{\psi \phi} A_d A_d^T - \frac{1}{\gamma^2} D D^T
\]
\[
- H_2 H_2^T - H_2 H_2^T - H_2 H_2^T > 0.
\]
Then, through applying the result of Peterson and Hollot (1986), one can easily show that, for any given positive-definite matrix \( Q_o \in \mathbb{R}^{12 \times 12} \) and any positive constants \( \psi \), \( \gamma \) satisfying (8), there exists a constant \( \epsilon > 0 \) such that, given any \( \epsilon \in (0, \epsilon^* \), the Riccati-like equation (7a) has a positive-definite solution matrix \( P \). Similarly, if there exists a constant \( \epsilon > 0 \) such that equation (7c) has a positive-definite solution matrix for some given positive constant \( \gamma \), some gain matrices \( K, F, G, R \) and \( S \), and a given positive-definite weighting matrix \( Q_o \), and if the following matrix inequality holds:
\[
C_1^T (I + \gamma^2 H_1 H_1^T)^{-1} C - \frac{1}{\gamma^2} K^T (I + N_1 N_1^T) K > 0. (9)
\]
then, one can also conclude that, for any given positive-definite matrix \( Q_o \in \mathbb{R}^{12 \times 12} \) and any positive constant \( \gamma \) satisfying the above inequality, there exists a constant \( \epsilon > 0 \) such that, given any \( \epsilon \in (0, \epsilon^* \), the Riccati-like equation (7c) has a positive-definite solution matrix \( P \).

**Remark 4.** By resorting to the standard optimal control theory, one can show that if the matrices \( A, B, H_1, H_2, H_3, D, \) and \( \gamma \) are in the same range space of \( B \), and (8) is satisfied and \((\gamma, B)\) is stabilizable, then the Riccati-like equation (7a) has a unique positive-definite solution matrix. The Riccati-like equation (7c) has a unique positive-definite solution matrix if the matrix \( K \) is in the range space of \( C_1 \) and (9) is satisfied and \( (\gamma, C_1) \) is stabilizable. It should be noted that, because \( (\gamma, C_1) \) is stabilizable, then the Riccati-like equation (7a) has a unique positive-definite solution matrix if the matrix \( K \) is in the range space of \( C_1 \) and (9) is satisfied and \( (\gamma, C_1) \) is stabilizable.

In view of the above Remarks 3 and 4, it is reasonable that we solve the Riccati-like equations (7a) and (7c) and obtain the gain matrices through the following design procedure:
(1) solve (7a) using the quadratic bound algorithm (Peterson and Hollot, 1986) by iterating \( \epsilon \);
(2) calculate the gain matrices \( K, F, G, R \) and \( S \);
(3) solve (7c) using the quadratic bound algorithm by iterating \( \gamma \);
(4) calculate the gain matrix \( L \).

For guaranteeing the success of the quadratic bound algorithm some structural restrictions on system matrices are required, but as discussed in Choi and Chung (1995a, b) the algorithm can still effectively be used to solve Riccati-like equations which do not satisfy such structural restrictions. And, many simulation experiences indicate that the failure or success of the above design algorithm is usually independent of the selection of the weighting matrices \( Q \), \( Q_o \), but it usually is required to choose \( \psi \) such that as many eigenvalues of the left-hand side of (8) as possible are positive big.

4. Example
Let \( \gamma = 1 \). Consider the system with
\[
A = \begin{bmatrix} -4 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 1 \\ 0.1 & 0 \end{bmatrix},
\]
\[
C = \begin{bmatrix} 1^T \\ 3^T \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1^T \\ 1 \end{bmatrix},
\]
\[
\Delta A = \begin{bmatrix} 0.1 \sin t & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta B = \begin{bmatrix} 0 \\ 0.1 \sin t \end{bmatrix},
\]
\[
\Delta A_d = \begin{bmatrix} 0.1 \sin t & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta C = \begin{bmatrix} 0.1 \sin t & 0 \end{bmatrix}^T.
\]
Take $F(t) = \sin t$, $N_1 = N_2 = N_3 = [1 \ 0]$, $N_4 = 1$, and

$$
H_1 = \begin{bmatrix} 0.1 & 0 \ 0 & 0.1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.1 & 0 \ 0 & 0.1 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0.1 & 0 \ 0 & 0.1 \end{bmatrix}, \quad H_4 = 0.1.
$$

With $Q_e = I$, $\psi_e = 2$ and $\varepsilon_0 = 0.0125$, the positive-definite solution of the Riccati-like equation (7a) is

$$
P_e = \begin{bmatrix} 0.012341 & 0.000387 \\
0.000387 & 0.014848 \end{bmatrix}
$$

and, therefore, by (6) the control parameters $K$, $F$, $G$, $R$ and $S$ are

$$
K = \begin{bmatrix} -0.5401 & 1.7973 \end{bmatrix}, \quad F = \begin{bmatrix} 0.9873 & 0.0310 \end{bmatrix},
$$

$$
G = \begin{bmatrix} 0.0015 & 0.0594 \end{bmatrix}, \quad R = \begin{bmatrix} 0.0031 \end{bmatrix},
$$

$$
S = \begin{bmatrix} 0.0987 \end{bmatrix}.
$$

With the above control parameters and $Q_0 = I$, $\psi_e = 2$, $\varepsilon_0 = 0.0125$, the positive-definite solution of the Riccati-like equation (7c) is

$$
P_0 = \begin{bmatrix} 0.006436 & 0.001584 \\
0.001584 & 0.014041 \end{bmatrix}
$$

and the control parameter $L$ is given by $L^T = \begin{bmatrix} -0.8862 \\
3.4620 \end{bmatrix}$. After all, the following control law is an $H_\infty$ controller guaranteeing the $H_\infty$ performance $\|H_\infty\| \leq 1$:

$$
\dot{x}(t) = \begin{bmatrix} -3.9356 & -3.4092 & 0.8862 \\
-5.0819 & -14.7601 & 3.4620 \end{bmatrix} x(t) + \begin{bmatrix} 0.8862 \\
3.4620 \end{bmatrix} y(t),
$$

$$
u(t) = -L \begin{bmatrix} 0.5401 \\
1.7973 \end{bmatrix} \dot{x}(t).
$$

5. Conclusion

In this note, we have given a design method for robust observer-based $H_\infty$ control of uncertain state delayed linear systems by extending the recent result of Choi and Chung (1996) and therefore the proposed method encompasses several special cases of interest. We have obtained a sufficient condition for guaranteeing not only the quadratic stability of the closed-loop system but also the $H_\infty$ norm bound constraint. According to the sufficient condition one can obtain control parameters of a robust observer-based $H_\infty$ control law by solving a pair of Riccati-like equations. Finally, we have given a simple design procedure to solve the Riccati-like equations.

References


