An LMI Approach to $H_\infty$ Controller Design for Linear Time-delay Systems*

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Abstract—We develop an output feedback $H_\infty$ control design method for linear time-delay systems based on the linear-matrix-inequality (LMI) approach. A sufficient condition for the existence of $H_\infty$ controllers of any order is given in terms of three LMIs. The condition is valid for both singular and regular problems. We also give a characterization of $H_\infty$ controllers in state space. The LMI existence condition can be easily determined whether or not, it is feasible and if it is then one can easily obtain $H_\infty$ controller gain matrices via convex optimization. © 1997 Elsevier Science Ltd.

1. Introduction
Many researchers have considered the problem of robust control by $H_\infty$ optimization. Recently, Doyle et al. (1989) derived simple state-space formulas for all controllers solving a standard $H_\infty$ problem: for a given $\gamma > 0$, find all controllers such that the $H_\infty$ norm of the closed-loop transfer function is less than $\gamma$. And various related works for linear systems have been reported (see e.g. Petersen, 1987, 1989; Gahinet and Apkarian, 1994; Iwatsuki and Skelton, 1994; Zhou et al. 1996). But, the problem of $H_\infty$ controller design for systems with time delay has rarely been considered. Recently, Choi and Chung (1996) propose two alternative methods for designing observer-based $H_\infty$ control laws whose gain matrices are obtained in terms of three LMIs, which define a convex set of solutions. Therefore, one can easily test whether the LMIs are solvable, and if so, one can get solutions yielding $H_\infty$ control parameters.

2. Problem formulation and background results
Let the system to be controlled be represented by the following differential equation:

$$\dot{x}(t) = Ax(t) + A_2x(t-h) + B_1w(t) + B_2u(t),$$

$$z(t) = C_1x(t) + D_{11}w(t) + D_{12}u(t),$$

$$y(t) = C_2x(t) + D_{21}w(t) + D_{22}u(t),$$

$$x(t) = \psi(t), \quad t \in [-h, 0].$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $y \in \mathbb{R}^r$ is the measured output, $w(t) \in \mathbb{R}^q$ is the square-integrable disturbance input, $z \in \mathbb{R}^s$ is the controlled output, $\psi(t) \in \mathbb{R}^s$ is the continuous initial value function, and $A, A_2, B_1, B_2$ are constant matrices with appropriate dimensions. It is assumed that $D_{22} = 0$ and that the triple $(A, B_2, C_2)$ is stabilizable and detectable. It should be noted that the assumption of $D_{22} = 0$ involves no loss of generality, while considerably simplifying algebraic manipulations (Gahinet and Apkarian, 1994; Iwatsuki and Skelton, 1994). Let the control system be the following output feedback compensator:

$$\dot{\hat{v}}(t) = K_{11}\hat{y}(t) + K_{12}w(t),$$

$$u(t) = K_{11}y(t) + K_{12}w(t),$$

where $K_{11}, K_{12}, K_{21}$ and $K_{22}$ have appropriate dimensions and (2) is a dynamic compensator of order $s$, $0 \leq s \leq n$. The extreme case $s = 0$ represents static gain output feedback. By introducing the augmented state vector $\tilde{x} = \begin{bmatrix} x^T & v^T \end{bmatrix}$, we can obtain the following closed-loop system:

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + A_2E\tilde{x}(t-h) + Bw(t),$$

$$z(t) = C\tilde{x}(t) + Dw(t),$$

$$\tilde{A} = \tilde{A} + \tilde{B}_2K\tilde{C}_2, \quad \tilde{B} = \tilde{B}_1 + \tilde{B}_2K\tilde{D}_2,$$

$$\tilde{C} = \tilde{C}_1 + \tilde{D}_{12}K\tilde{C}_2,$$

$$\tilde{D} = D_{11} + \tilde{D}_{12}K\tilde{D}_2, \quad \tilde{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E^T = \begin{bmatrix} I \\ 0 \end{bmatrix},$$

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix},$$

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} B_2 \\ 0 \end{bmatrix},$$

$$\tilde{C}_1 = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{C}_2 = \begin{bmatrix} C_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{D}_{12} = \begin{bmatrix} D_{12} \\ 0 \end{bmatrix},$$

Then the closed-loop transfer function $H_\infty(s)$ from $w$ to $z$ is given by

$$H_\infty(s) = \tilde{C}(sI - \tilde{A} - \tilde{\tilde{L}}E^{-sp})^{-1}\tilde{B} + \tilde{D}.$$

Thus our design problem is formulated as parameterizing...
the control gain matrices $K_{11}$, $K_{12}$, $K_{21}$, and $K_{22}$ of (2) such that the closed-loop control system of (1) and (2) is stable for some prescribed positive $\gamma$ the $H_\infty$ performance bound constraint $\|H_{\infty}(\cdot)\|_\infty \leq \gamma$ is guaranteed.

Before proceeding further, we shall give some preliminary results.

**Lemma 1.** For given some positive constant $\gamma$ and for a positive-definite matrix $Y$ and some matrices $A$, $\Pi$, $\Phi$, $E$ and $D$ with appropriate dimensions, if there exists a positive-definite matrix $P$ satisfying the inequality

$$A'TP + PA + P \Pi A^{-1} + \Phi'T \Phi + E'TE + \frac{1}{\gamma} PDD'TP < 0$$

then the delayed system $\dot{x}(t) = Ax(t) + PBx(t-h)$, where $x$ is the state vector and $h$ is some arbitrary positive constant, is asymptotically stable, and the following inequality holds for all $\omega$:

$$D'(\omega j\omega - A' + \Phi e^{-j\omega h})^{-1}E'(\omega j\omega - A + \Phi e^{-j\omega h})^{-1}D \leq \gamma^2.$$  


**Lemma 2.** The linear matrix inequality

$$Q(x) \preceq Y(x),$$

where $Q(x) = Q(x)'$, $R(x) = R(x)'$ and $S(x)$ depend affinely on $x$, is equivalent to

$$R(x) > 0, Q(x) - S(x)R(x)^{-1}S(x) > 0$$

**Proof.** See Albert (1992).

**Lemma 3.** Let $\Gamma > 0$.

$$\Psi(\cdot) - [B'(\omega j\omega - A')^{-1}] \Gamma [S + \Pi A^{-1}B]' + \Pi A^{-1}B] < 0$$

where $\Psi(\cdot)$ is the function of $x$ and $s$. Then the above inequality is solvable for $\Gamma$ if and only if

$$(\omega j\omega - A')^{-1}B > 0, (\omega j\omega - A')^{-1}B > 0.$$  

**Proof.** See Zhou et al. (1996).

**Lemma 4.** Given a symmetric matrix $\Omega$ and two matrices $\Gamma$ and $\Sigma$ with appropriate dimensions, consider the problem of finding some matrix $K$ such that

$$\Omega + \Gamma \Sigma \Sigma^T + \Sigma \Omega K < 0.$$  

Denote the orthogonal complements of $\Gamma$ and $\Sigma$ by $\bar{\Gamma}$ and $\bar{\Sigma}$ respectively. Then the above inequality is solvable for $K$ if and only if

$$\bar{\Omega} \bar{\Omega}^T < 0, \bar{\Sigma} \bar{\Sigma}^T < 0.$$  

**Proof.** See Gahinet and Apkarian (1994).

**3. Main results**

Consider the following matrix inequality:

$$A'TP + PA + PB^T \begin{bmatrix} C & P_{A1} \end{bmatrix} C^T + P_{Ax} + \frac{1}{\gamma} PDD'TP < 0,$$  

where $\Delta$ is an $n \times n$ positive-definite matrix and

$$\Delta = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}.$$  

Then we can establish the following.

**Proposition.** Suppose that for given generalized system (1) and controller (2), there exists a positive-definite matrix $P$ satisfying (5). Then the closed-loop control system response under the condition of zero disturbance input is asymptotically stable, i.e. $\dot{x}(t) = Ax(t) + Bx(t-h) + \dot{E}(t)$ is asymptotically stable, and the $H_\infty$ performance bound constraint $\|H_{\infty}(\cdot)\|_\infty \leq \gamma$ is satisfied.

**Proof.** By Lemma 2, we can see that (5) is equivalent to

$$\tilde{X}^T P + PA + \frac{1}{\gamma} \begin{bmatrix} \tilde{B}^T \tilde{C} & \tilde{B}^T \tilde{D} \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} \tilde{C}^T \tilde{C} \end{bmatrix}$$

$$+ \frac{1}{\gamma} \begin{bmatrix} \tilde{C}^T \tilde{D} \end{bmatrix} + PDD'TP < 0.$$  

And we can straightforwardly deduce that if (6) holds then $\tilde{X}(t) = \tilde{A}x(t) + \tilde{A}_1 \tilde{E}(t-h)$ is asymptotically stable. By Lemma 1, we can see that if (6a) holds and there exists a positive-definite matrix $P$ satisfying (6b) then the following inequality holds:

$$R'y_1 - [\omega j\omega - A' + \Phi e^{-\omega h}]^{-1}(\omega j\omega - A + \Phi e^{-\omega h})^{-1}D \leq \gamma^2.$$  

The above proposition is an extension of the bounded real lemma to linear time-delay systems. Now, by resorting to the above proposition, we shall get a sufficient condition for the existence of $H_\infty$ controllers. Using the expressions (3), we can rewrite (5) as follows:

$$\begin{bmatrix} \tilde{X}' \end{bmatrix} = A\begin{bmatrix} \tilde{X} \\ \tilde{E} \end{bmatrix} + B\begin{bmatrix} \tilde{E} \\ \tilde{E} \end{bmatrix} - A_1 \tilde{E}(t-h) + \tilde{D} \tilde{E}(t-h),$$

$$\begin{bmatrix} \tilde{X}' \end{bmatrix} = A\begin{bmatrix} \tilde{X} \\ \tilde{E} \end{bmatrix} + B\begin{bmatrix} \tilde{E} \\ \tilde{E} \end{bmatrix} - A_1 \tilde{E}(t-h) + \tilde{D} \tilde{E}(t-h),$$

$$\begin{bmatrix} \tilde{X}' \end{bmatrix} = A\begin{bmatrix} \tilde{X} \\ \tilde{E} \end{bmatrix} + B\begin{bmatrix} \tilde{E} \\ \tilde{E} \end{bmatrix} - A_1 \tilde{E}(t-h) + \tilde{D} \tilde{E}(t-h),$$

$$\begin{bmatrix} \tilde{X}' \end{bmatrix} = A\begin{bmatrix} \tilde{X} \\ \tilde{E} \end{bmatrix} + B\begin{bmatrix} \tilde{E} \\ \tilde{E} \end{bmatrix} - A_1 \tilde{E}(t-h) + \tilde{D} \tilde{E}(t-h).$$

By Lemma 4, we can obtain the following equivalent conditions where the control parameter $K$ is eliminated:

$$\Omega + \Gamma' K \Sigma \Sigma^T + \Sigma K \Gamma < 0.$$  

And, along similar lines to Gahinet and Apkarian (1994), we can obtain the following simplified equivalent conditions by utilizing the internal structure of the augmented matrices.
where $[W^T \ W^T]^T$ and $[V^T \ V^T]^T$ are any bases of the null spaces of $[B_1^T \ D_{11}^T]$ and $[C_2 \ D_{12}^T]$ respectively. We are now ready to state our main result.

**Main theorem.** Considering the closed-loop control system of (1) with (2), there exists a positive-definite matrix $P$ and a control gain matrix $K$ satisfying (5) if and only if there exist symmetric matrices $X$ and $Y$ satisfying (9). Therefore if there exist symmetric solution matrices to (9) then the closed-loop system response under the condition of zero disturbance input is asymptotically stable and the $H_\infty$ performance bound is satisfied. Moreover, if $\text{Rank} \ (I - XY) = k < n$ for solution matrices $X$ and $Y$ then there exist reduced-order $H_\infty$ controllers of order $k$.

Since (9) is affine in $X$ and $Y$ for fixed $y$ and $\Delta$, (9) defines a convex solution set of pairs $(X, Y)$, and therefore by using various efficient convex optimization algorithms one can easily test whether the LMIs are solvable, and if then one can obtain particular solutions yielding $H_\infty$ control parameters. In order to construct an $H_\infty$ controller, we first compute some solution $(X, Y)$ of the LMIs (9) using a convex optimization algorithm for some $y$ and $\Delta$. If $k = \text{Rank} \ (I - XY) = 0$ then we set $P = Y$. Otherwise, using the matrices $M$ and $N$ with two full column rank matrices $M$, $N \in \mathbb{R}^{n \times n}$ such that $$MN^T = I - XY,$$ we obtain the unique solution $P$ to the equation $$[X \ Y]^T P [X \ Y] = \begin{bmatrix} I & 0 \ 0 & 0 \ \end{bmatrix} = P^T [X \ Y]^T P.$$ It should be noted that this equation is always solvable if $X > \Theta$ and $M$ has full column rank (Gahinet and Apkarian, 1994). The solution matrix $P$ must satisfy (8), which is an LMI in the control parameter $K$; after all, one can easily compute control parameters for given $P$ by using a convex optimization algorithm. In summary, $H_\infty$ controllers can be obtained from feasible pairs $(X, Y)$ of (9) by solving the LMI (8) for $K$ through convex optimization. Alternatively, one can use more efficient explicit schemes. All control parameters satisfying the LMI (8) are parameterized explicitly as in Iwasaki and Skelton (1994) or Gahinet and Apkarian (1994). An explicit description of all solutions of the LMI (8) can be given as follows in state space:

$$K = -\rho \Sigma \Xi \Gamma (\Xi \Gamma^T)^{-1} + S \Sigma L (\Xi \Gamma^T)^{-1} \Sigma,$$\hspace{1cm} \text{(10)}

where $\Sigma$ and $L$ are free parameters subject to

$$\Xi = \left( \Sigma \Sigma^T - \frac{1}{\rho} I \right)^{-1} > \Theta, \quad \| L \| \leq \rho,$$\hspace{1cm} \text{(11)}

and the matrix $S$ is defined by

$$S \equiv I - \Xi \Gamma (\Xi \Gamma^T)^{-1} \Xi \Gamma.$$\hspace{1cm} \text{(12)}

Therefore, by implementing the explicit formulas, one can get the control parameters. Finally, we point out that the weighting matrix $\Delta$ constitutes an extra degree of freedom, which can be used to improve the design.

**Example.** Let $y = 1$ and $\Delta = I$. Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.1 & 0.1 \\ 1 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} 0.5 \\ 0 \\ \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}.$$\hspace{1cm} \text{(13)}

The following pair $(X, Y)$ satisfies (9):

$$X = \begin{bmatrix} 0.1053 & -0.1053 \\ -0.1053 & 0.2106 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}.$$\hspace{1cm} \text{(14)}

Since $\text{Rank} \ (I - XY) = 0$, we can see that there exist static output feedback $H_\infty$ control laws. If we set $\rho = 100$ then $\Xi$ of (11) is positive-definite, and therefore from (10) we can see that every static output feedback controller of the following parameterization guarantees not only asymptotic stability but also the $H_\infty$ performance constraint $\| H\sigma \| \leq 1$:

$$u = -(1.4890 - 0.0791)y,$$

where $l$ is any constant such that $| l | < 100$. And from the above parameterization, we can select the best by imposing additional constraints or criteria.

4. Conclusions

The problem of designing output feedback $H_\infty$ controllers for linear dynamic time-delay systems has been considered based on the linear-matrix-inequality (LMI) approach. We have given a sufficient condition for the existence of $H_\infty$ controllers of any order in terms of three LMIs defining a convex set of solutions. The control parameters can be parameterized explicitly in state space. The proposed method is an extension of the work of Gahinet and Apkarian (1994) and Iwasaki and Skelton (1994) to linear time-delay systems, and therefore it yields some design flexibility, and through some slight modification the earlier results can be reproduced.

**References**


